Solving Problems with Exponents and Logarithms

These notes are intended as a summary of section 5.8 (p. 430 – 434) in your workbook. You should also read the section for more complete explanations and additional examples.

Exponential and logarithmic functions are used to model many situations. The table below gives a number of examples.

Exponential Growth	Exponential Decay	Logarithmic Scales
• compound interest • population growth	• population decay • half-life of radioactive substances • amount of a drug in your body over time	• decibel scale • Richter scale • pH scale

Compound Interest

When money is invested in a savings account, it earns interest. If the interest is reinvested in the account, it also earns interest. This is called **compound interest**.

If a principal of P dollars is invested at an annual interest rate r (as a decimal), compounded n times per year, the amount, A, of the investment after t years is given by:

$$A = P \cdot \left(1 + \frac{r}{n}\right)^{nt}$$

Previously, we solved these problems using our graphing calculators. Now that we have learned to solve exponential and logarithmic equations, we can do them algebraically.

Example (not in workbook)
Determine the final value of $500 after 10 years, if it is invested at a rate of 2.25% and compounded monthly.

Example (not in workbook)
Determine the time it would take for an investment of $30 to double if it is compounded monthly at a rate of 7%.

Future Value

When a series of equal investments is made at regular time intervals, and when the compounding interval is equal to the time interval for the investments, the future value of the investment can be determined using

$$FV = \frac{R\left[(1+i)^n - 1\right]}{i}$$

where R is the regular investment (in dollars), i is the interest rate per compounding period, and n is the number of investments.

Example 1 (sidebar p. 431)
Determine how many monthly investments of \$200 would have to be made into an account that pays 6% annual interest, compounded monthly, for the future value to be \$100 000.

Present Value

It is common to borrow money to finance purchases of things like cars and homes. Such loans are usually repaid by making regular payments for a fixed amount of time. The amount borrowed is called the present value of the loan.

The following formula relates the present value (PV) to n equal payments of R dollars each, when the interest rate per compounding period is i.

$$PV = \frac{R\left[1-\left(1+i\right)^{-n}\right]}{i}$$

Note: The first payment is typically made after a time equal to the compounding period.

Example 2 (sidebar p. 432)
A person borrows $15 000 to buy a car. The person can afford to pay $300 a month. The loan will be repaid with equal monthly payments at 6% annual interest, compounded monthly. How many monthly payments will the person make?

Population Growth / Decay

Populations do not increase or decrease in jumps at the end of each year. Instead, they are continuously changing. This type of continuous growth / decay can be modelled by the function

$$A = A_0 \cdot e^{kt}$$

where A is the population size at time t, A_0 is the initial population size, and k is a constant that represents the growth rate.

Note: If k is positive, then the population is growing. If k is negative, then the population is decaying.

Example (not in workbook)
A radioactive substance is decaying according to the following formula:

$$A = A_0 \cdot e^{-0.2t}$$

where A is the amount of material remaining after t years, and A_0 is the initial amount of material.

a) If there was 80 g of the substance initially, how much is left after 3 years?

b) What is the half-life of the substance?

Logarithmic Scales

The **Richter scale**, decibel scale, and pH scale are all examples of logarithmic scales. Such scales are based on powers of 10, rather than a standard linear scale.

The Richter scale, for example, defines the magnitude, M, of an earthquake to be

$$M = \log\left(\frac{I}{S}\right)$$

where I is the intensity of an earthquake, measured 100 km from the epicenter, and S is the intensity of a standard earthquake (which has a seismograph reading of 1 micron).

Since this scale is logarithmic, an increase of 1 unit of magnitude represents a 10x increase in intensity. In other words, a magnitude 9 earthquake is 100 times more intense than a magnitude 7 earthquake.

Example 3 (sidebar p. 433)
The most intense earthquake ever recorded was in Chile in May 1960, with a magnitude of 9.5.

a) Calculate the intensity of the earthquake in Chile in terms of a standard earthquake.

b) How many times more intense was the Chile earthquake than the Haiti earthquake of 2010 (magnitude 7)? Give the answer to the nearest whole number.

Homework: #3 – 7, 9 – 12 in the section 5.8 exercises (p. 435 – 439). Answers on p. 440.